Magnetohydrodynamic pipe flow. Part 1

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The solution is obtained to the problem of the steady one-dimensional flow of an incompressible, viscous, electrically conducting fluid through a circular pipe in the presence of an applied (transverse) uniform magnetic field. A no-slip condition on the velocity is assumed at the non-conducting wall. The solution is exact and thus valid for all values of the Hartmann number. Excellent agreement exists between the present theoretical results and the experimental values obtained by Hartmann & Lazarus (1937) in the low to medium Hartmann number range. The high Hartmann number case is treated by Shercliff (1962) in the following paper.

1. Introduction

Hartmann, in his well-known paper (1937), considered the flow between two parallel, non-conducting walls with the applied magnetic field normal to the walls. An exact solution was obtained in this case because, with the exception of the pressure, all physical quantities depend only on the transverse co-ordinate. Shercliff (1953) solved the corresponding more general problem of the flow in a rectangular duct in which case both co-ordinates normal to the direction of fluid motion appear. His exact solution again demonstrated the fact that for large values of the Hartmann number, the velocity distribution consists of a uniform core with a boundary layer near the walls. It was this result which enabled him to solve the corresponding problem for a circular pipe in an approximate manner for large Hartmann numbers assuming walls of zero conductivity and, subsequently, walls with small conductivity (Shercliff 1956). Chang & Lundgren (1961) considered the effect of wall conductivity for this problem.

The present paper considers the problem of the steady one-dimensional (in the sense that only one component of the velocity and induced magnetic field are assumed to exist) flow of an incompressible, viscous, electrically conducting fluid through a circular pipe in the presence of a uniform transverse field. A no-slip condition on the velocity is assumed at the non-conducting wall. The flow is along the z-axis which coincides with the axis of the cylinder, and the applied magnetic field is along the x-axis, uniform and normal to the flow. The solution is exact and thus is valid for all values of the Hartmann number.

2. Basic equations

The motion of an electrically conducting fluid in the presence of a magnetic field obeys the well-known equations of magnetohydrodynamics. The fluid is treated as a continuum, and the classical results of fluid dynamics and electrodynamics are combined to express the phenomenon. For the steady flow of a viscous, incompressible fluid with constant properties, the full magnetohydrodynamic system can be reduced to just two equations involving the velocity, pressure, and magnetic field, i.e. the modified Navier–Stokes equation and the induction equation, along with the solenoidal conditions on the two vector quantities:

$$\rho(\mathbf{v}.\nabla)\mathbf{v}-(\mu/4\pi)(\mathbf{H}.\nabla)\mathbf{H} = -\nabla(p+\mu\mathbf{H}^2/8\pi)+\eta\nabla^2\mathbf{v}, \tag{1}$$

$$(4\pi\mu\sigma)\nabla\times(\mathbf{v}\times\mathbf{H})+\nabla^2\mathbf{H} = 0, \tag{2}$$

$$\nabla.\mathbf{v} = 0, \quad \nabla.\mathbf{H} = 0, \tag{3, 4}$$

where ρ, p, η, μ, $\mathbf{v}$, and σ are the fluid density, pressure, viscosity, permeability, velocity, and electrical conductivity, respectively, $\mathbf{H}$ is the magnetic field strength, and absolute electromagnetic units are used.

In the present one-dimensional problem, it is consistent with both the governing equations and the boundary conditions to assume that there is only one component of the velocity, v_z, and only one component of the induced magnetic field, H_z, along with the applied field H_0, so that the total velocity and magnetic field are given by

$$\left.\begin{aligned} v_r = v_\theta = 0, \quad v_z = v_z(r,\theta); \\ H_r = H_0\cos\theta, \quad H_\theta = -H_0\sin\theta, \quad H_z = H_z(r,\theta), \end{aligned}\right\} \tag{5}$$

where, except for the pressure, $\partial(\ )/\partial z = 0$. Substituting these expressions into (1), using cylindrical-polar co-ordinates, we obtain

$$p(r,\theta,z) = -(\mu/8\pi)H_z^2+K_1z+K_2, \quad \partial p/\partial z = \text{const.} = K_1, \tag{6}$$

$$K_1 = \eta\left[\frac{\partial^2 v_z}{\partial r^2}+\left(\frac{1}{r}\right)\frac{\partial v_z}{\partial r}+\left(\frac{1}{r^2}\right)\frac{\partial^2 v_z}{\partial\theta^2}\right]+\left(\frac{\mu}{4\pi r}\right)H_\theta\frac{\partial H_z}{\partial\theta}+\left(\frac{\mu}{4\pi}\right)H_r\frac{\partial H_z}{\partial r}, \tag{7}$$

where H_r and H_θ are given in equation (5). Equations (3) and (4) are identically satisfied, while equation (2) becomes

$$\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right)+\left(\frac{1}{r}\right)\frac{\partial^2 H_z}{\partial\theta^2}+4\pi\sigma\mu\left[H_r\frac{\partial}{\partial r}(rv_z)+\frac{\partial}{\partial\theta}(v_zH_\theta)\right] = 0. \tag{8}$$

Equations (6) through (8) define the variables $p(r,\theta,z)$, $v_z(r,\theta)$, and $H_z(r,\theta)$ subject to the following homogeneous boundary conditions. There is no fluid slip at the wall, hence

$$v_z(a,\theta) = 0, \tag{9}$$

where a is the radius of the cylinder, while the assumption of non-conducting walls implies that (see Shercliff 1953)

$$H_z(a,\theta) = 0. \tag{10}$$

When (6) through (10) have been solved for p, v_z, and H_z, we can obtain the current density $\mathbf{J}$ and the electric field strength $\mathbf{E}$ from Ampère's law and Ohm's law, respectively

$$j_r = \left(\frac{1}{4\pi r}\right)\frac{\partial H_z}{\partial\theta}, \quad j_\theta = -\left(\frac{1}{4\pi}\right)\frac{\partial H_z}{\partial r}, \quad j_z = 0, \tag{11}$$

$$E_r = (1/\sigma)j_r+\mu v_zH_\theta, \quad E_\theta = (1/\sigma)j_\theta-\mu v_zH_r, \quad j_z = 0. \tag{12}$$

Let us now introduce the non-dimensional variables $v = v_z/v_0$, $H = H_z/H_0$, $\rho = r/a$, where v_0 is some characteristic velocity, a is the radius of the pipe, and H_0 is the uniform applied transverse field. Equations (7) and (8) take on the following form in $v(\rho,\theta)$ and $H(\rho,\theta)$

$$\nabla^2 v - \left(\frac{M^2}{R_M}\right)\left[\left(\frac{\sin\theta}{\rho}\right)\frac{\partial H}{\partial\theta} - \cos\theta\,\frac{\partial H}{\partial\rho}\right] = K, \tag{13}$$

$$\nabla^2 H - R_M\left[\left(\frac{\sin\theta}{\rho}\right)\frac{\partial v}{\partial\theta} - \cos\theta\,\frac{\partial v}{\partial\rho}\right] = 0, \tag{14}$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial\rho^2} + \left(\frac{1}{\rho}\right)\frac{\partial}{\partial\rho} + \left(\frac{1}{\rho^2}\right)\frac{\partial^2}{\partial\theta^2}, \quad M = \mu H_0 a(\sigma/\eta)^{\frac{1}{2}}$$

is the Hartmann number, $R_M = 4\pi\sigma\mu v_0 a$ is the magnetic Reynolds number, and $K = K_1 a^2/v_0\eta$.

3. Exact solution

It is possible to eliminate the first-order derivatives in such equations by the customary procedure of introducing an exponential factor $\exp[(\text{const.})x]$. At the same time, the equations may be readily uncoupled by a linear transformation. It follows that the substitutions

$$f(\rho,\theta) = e^{\alpha\rho\cos\theta}\left[v + \left(\frac{2\alpha}{R_M}\right)H - \frac{K\rho\cos\theta}{2\alpha}\right], \tag{15}$$

$$g(\rho,\theta) = e^{-\alpha\rho\cos\theta}\left[v - \left(\frac{2\alpha}{R_M}\right)H + \frac{K\rho\cos\theta}{2\alpha}\right], \tag{16}$$

transform equations (13) and (14) into the simplified forms

$$\nabla^2 f - \alpha^2 f = 0, \quad \nabla^2 g - \alpha^2 g = 0, \tag{17, 18}$$

where $\alpha = \frac{1}{2}M$. It is perhaps noteworthy to point out that the above discussion is restricted only by the assumptions of one-dimensionality, no variation of the physical quantities (except for the pressure) in the flow direction, and a uniform applied transverse field. Thus, equations (17) and (18) apply to any such general magnetohydrodynamic (incompressible, steady) duct flow. The restriction as to geometry and the conditions at the wall will enter through the boundary conditions. In the present problem (9) and (10) reduce to the following non-homogeneous boundary conditions on f and g

$$f(1,\theta) = -(K/2\alpha)\cos\theta\, e^{\alpha\cos\theta}, \quad g(1,\theta) = (K/2\alpha)\cos\theta\, e^{-\alpha\cos\theta}. \tag{19}$$

The solution of equations (17) through (19) has been obtained using a Fourier analysis (Tanazawa 1960; Fabri & Siestrunck 1960; Uflyand 1960; Gold 1961; Uhlenbusch & Fischer 1961). The velocity and magnetic-field distributions as presented in the author's paper are given by

$$v = \frac{-K}{4\alpha}\left[e^{-\alpha\rho\cos\theta}\sum_{n=0}^{\infty}\epsilon_n\frac{I_n'(\alpha)}{I_n(\alpha)}I_n(\alpha\rho)\cos n\theta + e^{\alpha\rho\cos\theta}\sum_{n=0}^{\infty}(-1)^n\epsilon_n\frac{I_n'(\alpha)}{I_n(\alpha)}I_n(\alpha\rho)\cos n\theta\right], \tag{20a}$$

or
$$v = \frac{-K}{2\alpha}\left[\cosh(\alpha\rho\cos\theta)\sum_{n=0}^{\infty}\epsilon_n\frac{I'_{2n}(\alpha)}{I_{2n}(\alpha)}I_{2n}(\alpha\rho)\cos 2n\theta - \sinh(\alpha\rho\cos\theta)\sum_{n=0}^{\infty}2\frac{I'_{2n+1}(\alpha)}{I_{2n+1}(\alpha)}I_{2n+1}(\alpha\rho)\cos(2n+1)\theta\right], \quad (20b)$$

where I_n is the modified Bessel function of order n, $\epsilon_n = 1$ for $n = 0$, $\epsilon_n = 2$ for $n > 0$; and the minus sign reflects the fact that v is opposite in sign to the pressure-gradient term in K, and

$$H = \frac{-R_M K}{8\alpha^2}\left[e^{-\alpha\rho\cos\theta}\sum_{n=0}^{\infty}\epsilon_n\frac{I'_n(\alpha)}{I_n(\alpha)}I_n(\alpha\rho)\cos n\theta - e^{\alpha\rho\cos\theta}\sum_{n=0}^{\infty}(-1)^n\epsilon_n\frac{I'_n(\alpha)}{I_n(\alpha)}I_n(\alpha\rho)\cos n\theta - 2\rho\cos\theta\right], \quad (21a)$$

or
$$H = \frac{-R_M K}{4\alpha^2}\left[\cosh(\alpha\rho\cos\theta)\sum_{n=0}^{\infty}\epsilon_n\frac{I'_{2n+1}(\alpha)}{I_{2n+1}(\alpha)}I_{2n+1}(\alpha\rho)\cos(2n+1)\theta - \sinh(\alpha\rho\cos\theta)\sum_{n=0}^{\infty}2\frac{I'_{2n}(\alpha)}{I_{2n}(\alpha)}I_{2n}(\alpha\rho)\cos 2n\theta - \rho\cos\theta\right]. \quad (21b)$$

Expanding (20*b*) for small values of α we obtain

$$\lim_{\alpha\to 0} v(\rho,\theta;\alpha) = -\tfrac{1}{4}K(1-\rho^2), \quad \text{or} \quad v_z|_{\alpha=0} = -\frac{(\partial p/\partial z)}{4\eta}(a^2-r^2),$$

which is the classical non-conducting result.

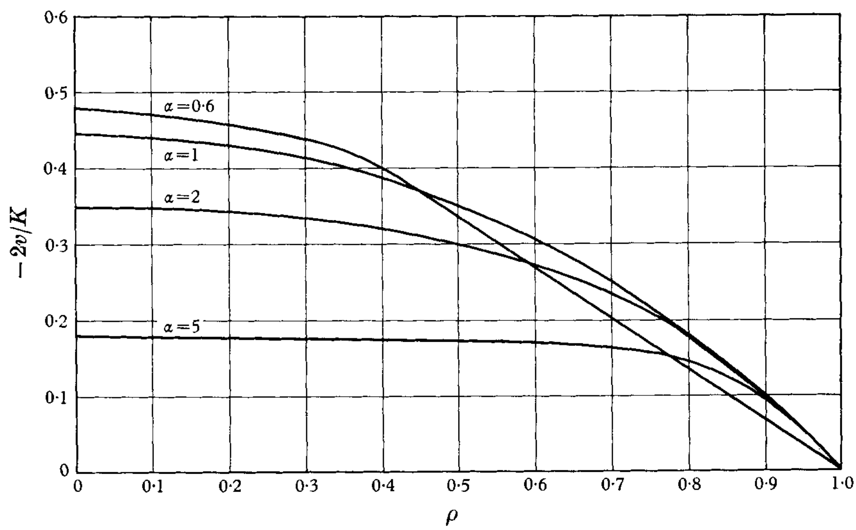

FIGURE 1. Velocity distribution in the pipe, $\theta = 0$.

The velocity and magnetic field profiles are plotted in figures 1 and 3 for $\theta = 0$ and for several values of the Hartmann number $M = 2\alpha$. As M increases, the uniform core of velocity is clearly demonstrated. Figure 2 shows the variation of v with ρ at $\theta = \frac{1}{2}\pi$ in which case the core-like character of the velocity distribution is less apparent. Note that $H(\rho, \frac{1}{2}\pi) = H(\rho, \frac{3}{2}\pi) = 0$ such that there is no current flow across the y-axis, $-1 \leqslant \rho \leqslant 1$. The following additional properties are readily obtained from (20) and (21)

$$v(\rho,\theta\pm\pi) = v(\rho,\theta), \qquad v(\rho,-\theta) = v(\rho,\theta),$$

$$H(\rho,\theta\pm\pi) = -H(\rho,\theta), \quad H(\rho,-\theta) = H(\rho,\theta),$$

$$v(0,\theta) = \frac{-K}{2\alpha}\frac{I_0'(\alpha)}{I_0(\alpha)} = \frac{-K}{2\alpha}\frac{I_1(\alpha)}{I_0(\alpha)}, \quad H(0,\theta) = 0.$$

Using equations (11) and (12) we obtain, in non-dimensional form,

$$E_r = \mu H_0 v_0 \left[\left(\frac{1}{R_M\rho}\right)\frac{\partial H}{\partial\theta} - v\sin\theta\right]. \tag{22}$$

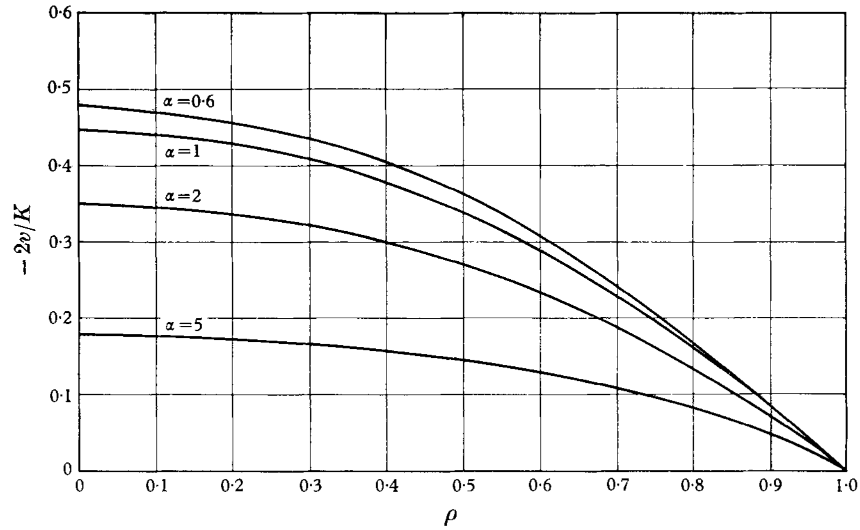

FIGURE 2. Velocity distribution in the pipe, $\theta = \frac{1}{2}\pi$.

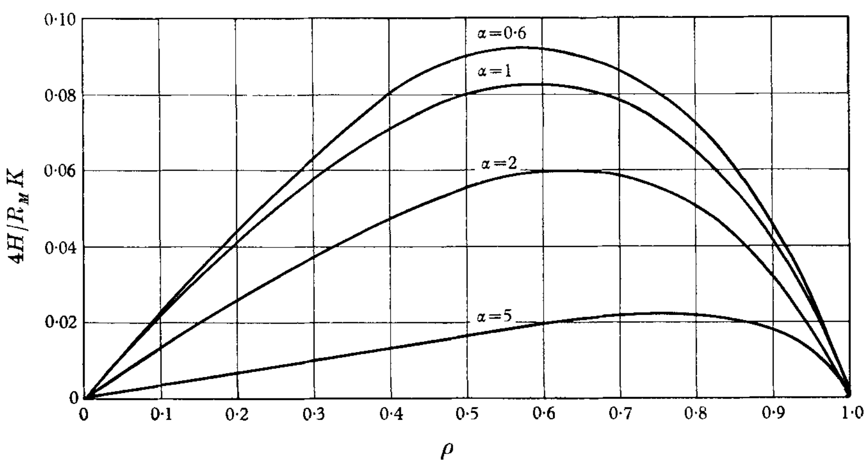

FIGURE 3. Induced magnetic field, $\theta = 0$.

The electric potential V_{ab} between the points $a = (1, \frac{1}{2}\pi)$ and $b = (1, -\frac{1}{2}\pi)$ induced by the motion is

$$V_{ab} = 2\int_{r=0}^{a} E_r(r, \tfrac{1}{2}\pi)\, dr = 2\mu H_0 v_0 a \int_{\rho=0}^{1} \left\{ \frac{1}{R_M \rho} \left[\frac{\partial}{\partial\theta} H(\rho,\theta) \right]_{\theta=\frac{1}{2}\pi} - v(\rho, \tfrac{1}{2}\pi) \right\} d\rho. \quad (23)$$

The sensitivity S, employed by Shercliff (1956), is thus given by the following expression which is valid for all α,

$$S = \frac{V_{ab}}{2\mu H_0 v_0 a} = \frac{K}{2\alpha}\int_{\rho=0}^{1} \left\{ \frac{1}{\alpha\rho} \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{I'_{2n+1}(\alpha)}{I_{2n+1}(\alpha)} I_{2n+1}(\alpha\rho) + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \epsilon_n \frac{I'_{2n}(\alpha)}{I_{2n}(\alpha)} I_{2n}(\alpha\rho) - \frac{1}{2\alpha} \right\} d\rho. \quad (24)$$

S is plotted *vs* M in figure 4.

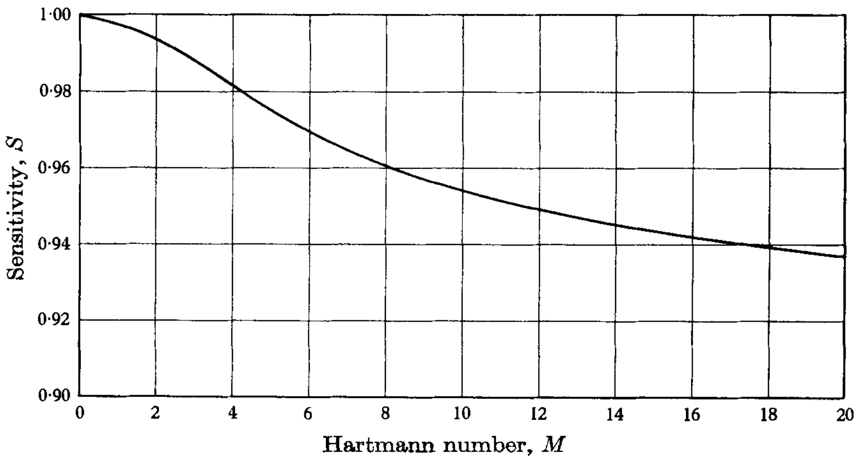

FIGURE 4. Sensitivity *vs* Hartmann number.

4. Comparison of present results with experimental data and Shercliff's large M approximate theory

In order to compare the present results with the experimental values given by Shercliff (1953), it is necessary to obtain the mean velocity by integrating the velocity distribution over the circular cross-section, i.e.

$$\bar{v}(\alpha) = \frac{1}{\pi}\int_{\rho=0}^{1}\int_{\theta=0}^{2\pi} \rho v(\rho,\theta)\, d\rho\, d\theta. \quad (25)$$

Using $(20a)$, the identities

$$I_n(\alpha\rho) \equiv \frac{(-1)^n}{2\pi}\int_{\theta=0}^{2\pi} e^{-\alpha\rho\cos\theta} \cos n\theta\, d\theta \equiv \frac{1}{2\pi}\int_{\theta=0}^{2\pi} e^{\alpha\rho\cos\theta}\cos n\theta\, d\theta,$$

and the expression

$$\int_{\rho=0}^{1} \rho I_n^2(\alpha\rho)\, d\rho = \frac{1}{2\alpha^2}(\alpha^2+n^2) I_n^2(\alpha) - \tfrac{1}{2} I_n'^2(\alpha),$$

we obtain for the mean velocity

$$\frac{-\bar{v}}{K} = \sum_{n=0}^{\infty} (-1)^n \epsilon_n \frac{I'_n(\alpha)}{2\alpha I_n(\alpha)} \left[\left(1+\frac{n^2}{\alpha^2}\right) I_n^2(\alpha) - I_n'^2(\alpha) \right]. \quad (26)$$

The following alternate forms are of interest

$$\frac{-\bar{v}}{K} = \sum_{n=0}^{\infty} (-1)^n \epsilon_n \frac{I_n'(\alpha)}{2\alpha I_n(\alpha)} \left[\frac{n^2}{\alpha^2} I_n^2(\alpha) - I_n'^2(\alpha)\right], \tag{26a}$$

and
$$\frac{-\bar{v}}{K} = \frac{-I_1^3(\alpha)}{2\alpha I_0(\alpha)} + \frac{1}{2\alpha}\sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{I_{n-1}(\alpha)+I_{n+1}(\alpha)}{I_n(\alpha)}\right] I_{n-1}(\alpha)\, I_{n+1}(\alpha), \tag{26b}$$

since
$$\sum_{n=0}^{\infty} (-1)^n \epsilon_n I_n(\alpha) I_n'(\alpha) \equiv 0, \quad (n/\alpha) I_n(\alpha) \equiv \tfrac{1}{2}[I_{n-1}(\alpha) - I_{n+1}(\alpha)],$$

and
$$I_n'(\alpha) \equiv \tfrac{1}{2}[I_{n-1}(\alpha) + I_{n+1}(\alpha)],$$

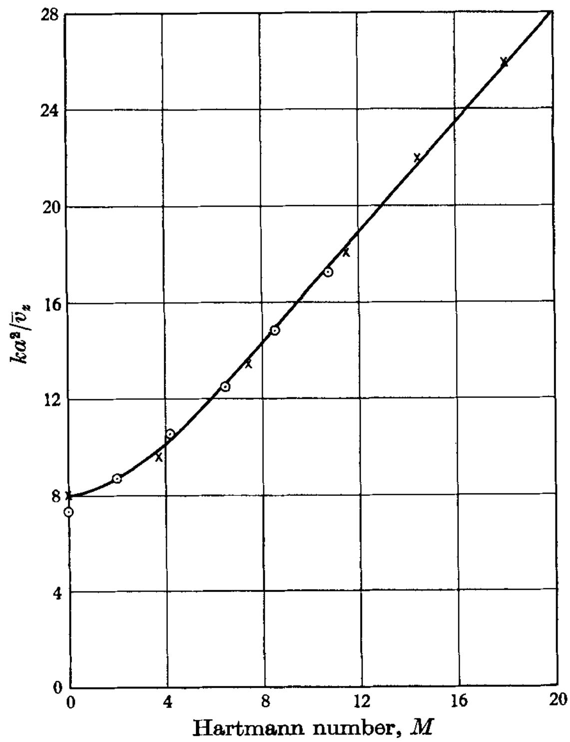

FIGURE 5. Mean velocity *vs* Hartmann number. —, Present theory; ⊙, ×, experimental data of Hartmann & Lazarus.

while additional representations may be derived by expressing $I_n'(\alpha)$ differently or by summing the series $n^2 I_n(\alpha) I_n'(\alpha)$ separately. Note that in dimensional form, $-\bar{v}/K = \bar{v}_z/ka^2$, where $(\partial p/\partial z) = -k\eta$ and $\bar{v}_z$ is the integrated dimensional velocity, v_0 in Shercliff's notation. The classical result must follow from (26) and, indeed,

$$\lim_{\alpha\to 0} \bar{v}_z(\alpha) = \tfrac{1}{8} ka^2 = -\frac{(\partial p/\partial z)\, a^2}{8\eta}.$$

In figure 5 the reciprocal relation $ka^2/\bar{v}_z$ is plotted versus the Hartmann number. The agreement between the present theory and the numerical data of Hartmann & Lazarus (1937) is clearly shown. In summing (26) for large values of α, numerical difficulties are encountered because of the appearance of small differences between very large numbers. As a result $ka^2/\bar{v}_z$ was not computed exactly in this range. An asymptotic expansion of the solution was sought in the following manner. Introducing a logarithmic substitution $u(\alpha) = \ln[I_n(\alpha)]$ into Bessel's equation we can derive an asymptotic representation for the function $u'(\alpha) = I_n'(\alpha)/I_n(\alpha)$. Substituting this result into the velocity distribution (20a) we obtain a $(1/M)$ correction to Shercliff's solution (1953)

$$\frac{-v}{K}\bigg|_{M\gg 1} = \frac{1}{M}(1-\rho^2\sin^2\theta)^{\frac{1}{2}}\left[1-\frac{1}{M(1-\rho^2\sin^2\theta)^{\frac{3}{2}}}\right], \tag{27}$$

with the corresponding integrated value

$$\frac{ka^2}{\bar{v}_z}\bigg|_{M\gg 1} = \frac{3\pi M}{8}\left(1+\frac{3\pi}{4M}\right). \tag{28}$$

By a similar calculation it follows, to first order, that

$$S_{M\gg 1} = \frac{K\pi}{8\alpha} = \frac{-ka^2\pi}{8\alpha\bar{v}_z} = \frac{-3\pi^2}{32} = -0{\cdot}925. \tag{29}$$

Although the series representations which lead to (27) diverge at $\rho^2\sin^2\theta = 1$, the mean velocity can still be derived since the corresponding integrals converge. A more serious formal consideration arises from the use of the asymptotic representation for $I_n'(\alpha)/I_n(\alpha)$. Being semiconvergent in that it is valid only for n sufficiently small compared with α (which is large but fixed), its use in the infinite sums in equation (20a) cannot be formally justified. Examination of the numerical convergence of the resultant infinite sums indicates that the contribution of the terms beyond the validity of the asymptotic expression is negligibly small (of the order $1/\alpha^2$). In the final analysis, however, observe that the $(1/M)$ correction in (27), like Shercliff's first-order result, states that v is a function of $y = r\sin\theta$ only and does not satisfy the boundary condition $v = 0$ at the wall (except at the points $y = \pm a$). Mathematically this may be related to the aforementioned divergence at $\rho^2\sin^2\theta = 1$. Physically, it is clear that (27) represents the core profile and the boundary-layer defect is not fully accounted for in equation (28).

The large Hartmann number case is treated in the following paper by Shercliff, thus extending his earlier analysis (1953). Accounting for the boundary-layer defect, for example, results in a refined value of $(3\pi/2M)$ in (28) which is in good agreement with his earlier experimental results (1956).

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